Exercise 2.5.2

("Blow-up": Reaching infinity in a finite time) Show that the solution to $\dot{x} = 1 + x^{10}$ escapes to $+\infty$ in a finite time, starting from any initial condition. (Hint: Don't try to find an exact solution; instead, compare the solutions to those of $\dot{x} = 1 + x^2$.)

Solution

$\dot{x}$ is the rate that $x(t)$ grows as t increases, so if the solution to

$$\dot{x} = 1 + x^2, \quad x(0) = x_0,$$

blows up in finite time, then the solution to

$$\dot{x} = 1 + x^{10} = 1 + (x^2)^5, \quad x(0) = x_0,$$

certainly will as well because $1 + x^{10} > 0$ for all x and $1 + x^{10} > 1 + x^2$ for $|x| > 1$. Solve the previous initial value problem by separating variables and integrating both sides.

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{dx}{1 + x^2} = dt$$

$$\int \frac{dx}{1 + x^2} = \int dt$$

$$\tan^{-1} x = t + C$$

Use the initial condition $x(0) = x_0$ to determine C.

$$\tan^{-1} x_0 = 0 + C \quad \to \quad C = \tan^{-1} x_0$$

So the previous equation becomes

$$\tan^{-1} x = t + \tan^{-1} x_0.$$

Therefore,

$$x(t) = \tan\left(t + \tan^{-1} x_0\right).$$

This solution blows up when

$$t + \tan^{-1} x_0 = \frac{\pi}{2} \quad \to \quad t = \frac{\pi}{2} - \tan^{-1} x_0,$$

so by comparison the solution to

$$\dot{x} = 1 + x^{10}, \quad x(0) = x_0$$

blows up in finite time too.